Math 210A Lecture 21 Notes

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1 Frattini's Argument and Characterizations of Nilpotent Groups

1.1 Frattini's argument

Theorem 1.1 (Frattini's argument)**.** *Let G be a finite group, $N \trianglelefteq G$, and let P be a Sylow p-subgroup of N. Then $G = NN_G(P)$.*

Proof. If $g \in G$, then $gPg^{-1} \leq N$ (since $N \trianglelefteq G$). So gPg^{-1} is Sylow p in N, and therefore, there exists some $n \in N$ such that $gPg^{-1}nPn^{-1}$. Then $n^{-1}g \in N_G(P)$. So $g \in NN_G(P)$. $\square$

1.2 Characterizations of nilpotent groups

Theorem 1.2. *Let G be a finite group. The following are equivalent:*

1. *G is nilpotent.*
2. *If $H < G$, then $H < N_G(H)$.*
3. *If $P \in \mathrm{Syl}_p$, then $P \trianglelefteq G$.*
4. *$G \cong \prod_{p \text{ prime}} P_p$, where P_p is a Sylow p-subgroup.*
5. *If $M < G$ is a maximal proper subgroup (not contained in any other proper subgroup), then $M \trianglelefteq G$.*

Proof. (1) $\implies$ (2): Suppose $N < G$. If $HZ(G) = G$¡ then $G = N_G(H)$, so $H < N_G(H)$. If $HZ(G) \neq G$, $N_G(HZ(G)) = N_G(H)$, so we may assume that $Z(G) \leq H$ (replace H by $HZ(G)$). Now $H/Z(G) < G/Z(G)$. If G has nilpotence class n, then $G/Z(G)$ has nilpotence class $\leq n-1$. By induction, $H/Z(G) < N_{G/Z(G)}(H/Z(G))$. This is $N_G(H)/Z(G)$, so $H < H_G(H)$.

(2) $\implies$ (3): If G is a p-group, then $G \trianglelefteq G$, so we are done. If G is not a p-group, let $P \in \mathrm{Syl}_p(G)$ with $P < G$. Then $P \trianglelefteq N = N_G(P)$, and $P < N$. P is unique of its order, so it is characteristic in N. So $P \trianglelefteq N_G(N)$. So $N = N_G(N)$. By (2), $N = G$. So $P \trianglelefteq G$.

(3) $\implies$ (4): This is the Krull-Schmidt theorem.

(4) $\implies$ (5): Let $M < G$ be maximal, and suppose that $p_1, \dots, p_s$ are the distinct primes dividing $|G|$. If $s = 1$, then Sylow's theorems give us a subgroup of order p^{n-1} normal in G, where $|G| = p^n$. If $s > 1$, let $P_1, \dots, P_s$ be our Sylow p-subgroups. For $M < G$ is maximal, we claim that there exists a unique i such that $M \cap P_i \neq P_i$. Existence is clear, and for uniqueness, $M < MP_i = G$, which forces $M \cap P_j = P_j$ for all $j \neq i$. Then $M \cong (M \cap P_i) \times \prod_{j \neq i} P_j$. Sylow's theorems imply that $M \cap P_i \trianglelefteq P_i$, so $M \trianglelefteq G$.

(5) $\implies$ (3): Let $P \in \mathrm{Syl}_p(G)$ with $P \ntrianglelefteq G$. Then $N_G(P) \leq M < G$, where M is maximal. Then $M \trianglelefteq G$, and $P \in \mathrm{Syl}_p(M)$. By Frattini's argument, $G = MN_G(P) = M$. This is a contradiction.

(4) $\implies$ (1): $G \cong \prod_{i=1}^s P_i$. Since p-groups are nilpotent, G is nilpotent. $\square$

Proposition 1.1. *Let G be nilpotent, and let $S \subseteq G$ with image generating $G^{\mathrm{ab}} = G/[G,G]$. Then S generates G.*

Proof. Proceed by induction on the nilpotence class n. If $n = 1$, then $G = G^{\mathrm{ab}}$. If $n \geq 2$, then $(G/G_n)^{\mathrm{ab}} \cong G/(G_n G_2) \cong G^{\mathrm{ab}}$. By induction, $\mathrm{im}(S)$ generates G/G_n. If $H = \langle S \rangle \leq G$, then $G = G_n H$. $G_n \leq Z(G)$, so $N_G(H) = G$. So $H \trianglelefteq G$. Then $G_n = [G_{n-1}, G] = [G_{n-1}, G_n H] = [G_{n-1}, H] \leq H$ (since $H \trianglelefteq G$). So $G = G_n H = H = \langle S \rangle$. $\square$

Theorem 1.3. *If p is prime, then there exist exactly 2 isomorphism classes of nonabelian groups of order p^3, represented by*

1. *if $p = 2$, D_4 and Q_8,*
2. *if p is odd,* $\mathrm{Heis}(\mathbb{Z}/p\mathbb{Z}) \cong (\mathbb{Z}/p^2\mathbb{Z})^2 \rtimes \mathbb{Z}/p\mathbb{Z}$ *and*

$$K = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z}/p^2\mathbb{Z}) : a \equiv 1 \mod p \right\} \cong \mathbb{Z}/p^2\mathbb{Z} \rtimes_\varphi \mathbb{Z}/p\mathbb{Z},$$

where $\varphi(1)$ is multiplication by $1 + p$.

Remark 1.1. $\mathrm{Heis}(\mathbb{Z}/2\mathbb{Z}) \cong D_4$. For p odd, $\mathrm{Heis}(\mathbb{Z}/p\mathbb{Z})$ has no elements of order p^2.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^p = \begin{bmatrix} 1 & p & \binom{p}{2} \\ 0 & 1 & p \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \binom{p}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1.3 Linear groups

Lemma 1.1.

$$|\operatorname{GL}_n(\mathbb{F}_q)| = (q^n - 1)(q^n - q)\cdots(q^n - q^{n-1}) = q^{n(n-1)/2}\prod_{i=1}^{n}(q^i - 1).$$

$$|\operatorname{SL}_n(\mathbb{F}_q)| = q^{n(n-1)/2}\prod_{i=2}^{n}(q^i - 1).$$

Proof. For the order of $\operatorname{GL}_n(\mathbb{F}_q)$, we have $q^n - 1$ choices for the first column, then $q^n - q$ choices for the second columns, etc. since the columns must be linearly independent.

For $\operatorname{SL}_n(\mathbb{F}_q)$, we quotient out by the determinant map, which is onto $\mathbb{F}_p^\times$. $\square$

Definition 1.1. The **projective special linear group** is $\operatorname{PSL}_n(F) = \operatorname{SL}_n(F)/Z(\operatorname{SL}_n(F))$.

Proposition 1.2.

$$\operatorname{SL}_n(F) = \langle\{E_{i,j}(\alpha) : \alpha \in F, i \neq j\}\rangle$$